Performance Evaluation of Decreasing Multi-State Consecutive-k-out-of-n:G Systems

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Abstract

In the binary context, a consecutive-k-out-of-n:G system works if and only if at least k consecutive components are working. In the multi-state context, a consecutive-k-out-of-n:G system is in state j or above ($j = 1, 2, \cdots, M$) if and only if at least k_l consecutive components are in state l or above for all l ($1 \leq l \leq j$). In this paper, we use minimal path vectors to evaluate the system state distribution. When $M = 3$, a recursive formula is provided for evaluating the system state distribution. When $M \geq 4$, an algorithm is provided to bound the system state distribution. These bounds are sharper than those reported in the literature.

1 Introduction

In traditional reliability theory, both the system and its components are allowed to take only two possible states: either working or failed. A system with n components is called a consecutive-k-out-of-n:F (G) system if it fails (works) whenever at least k consecutive components in the system fail (work). Many research results have been reported on reliability evaluation of binary consecutive-k-out-of-n:F and G systems, for example, see Hwang (1982) and Chao et al. (1995). The dual relationship between the consecutive-k-out-of-n:F and G systems is investigated by Kuo et al. (1990) and Zuo (1993).

In a multi-state system, both the system and the components are allowed to be in $M + 1$ possible states, $0, 1, 2, \ldots,$ and M, where M is the perfect state while 0 is the complete failure state. Lately a few researchers have extended the definitions of the binary consecutive-k-out-of-n system to the multi-state case by allowing the system to remain binary and its components to have more than two possible states, for example, see Zuo and Liang (1994), Kossow and Preuss (1995), and Malinowski and Preuss (1995). Huang et al. (2003) propose more general definitions of the multi-state consecutive-k-out-of-n:F and G systems. In their definitions, a possibly different number of consecutive components need to be below state j for the multi-state consecutive-k-out-of-n:F system to be below state j. They provide an algorithm for evaluating system state distribution of decreasing multi-state consecutive-k-out-of-n:F systems. Another algorithm is provided to bound system state distribution of increasing multi-state consecutive-k-out-of-n:F systems.

In this paper, we study the multi-state consecutive-k-out-of-n:G systems. Minimal path vectors are used for evaluating the system state distribution. Vector $\mathbf{y} \in \mathbf{S}^n$ is a minimal path vector to system state j if and only if $\phi(\mathbf{y}) \geq j$ and $\phi(\mathbf{x}) < j$ for all $\mathbf{x} < \mathbf{y}$ and vector $\mathbf{y} \in \mathbf{S}^n$ is a minimal cut vector to system state j if and only if $\phi(\mathbf{y}) < j$ and $\phi(\mathbf{x}) \geq j$ for all $\mathbf{x} > \mathbf{y}$ (Boedigheimer and Kapur, 1994). The system is a multi-state monotone system, that is, (1) $\phi(\mathbf{x})$ is non-decreasing in each argument; and (2) $\phi(\mathbf{j}) = \phi(j, j, \ldots, j) = j$ for $j = 0, 1, \cdots, M$ (Griffith 1980). We also assume that the x_i's are mutually s-independent.

Notation:

x_i, $\mathbf{x}$	state of component i, $x_i \in \{0, 1, \cdots, M\}$, $i = 1, 2, \ldots, n$; $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.
$\phi(\mathbf{x})$	system structure function representing the state of the system, $\phi(\mathbf{x}) \in \{0, 1, \ldots, M\}$.
k_j	minimum number of consecutive components to be in states below j
P_{ij}, P_j	$\Pr(x_i \geq j)$; $P_j = P_{ij}$ when the components are i.i.d.

p_{ij}, p_j	$\Pr(x_i = j)$; $p_j = p_{ij}$ when the components are i.i.d.
Q_{ij}, Q_j	$\Pr(x_i < j)$; $Q_j = Q_{ij}$ when the components are i.i.d.
$R_2(n, k_1, k_2)$	probability that at least k_l consecutive components are in state m_l or above for $l = 0, 1, 2$ in an n component system
R_{sj}, r_{sj}	$\Pr(\phi \geq j)$; $\Pr(\phi = j)$.
$R(n; k)$	Pr(at least k consecutive components are "working" and all other components are "failed")
$R(a, b)$	Pr(at least b consecutive components are in state m_1 or above among the first a components)
$R^{(n)}(a, b)$	Pr(at least b consecutive components are in state m_2 or above among the last a components)
$R'(a, b)$	Pr(at least b consecutive components are in state m_2 or above among the first a components)

2 The Multi-State Consecutive-k-out-of-n:G System

In the binary context, a consecutive-k-out-of-n:G system works if and only if at least k consecutive components work (Kuo et al. 1990). Huang et al. (2003) propose the following definition of the multi-state consecutive-k-out-of-n:G system.

Definition 1 (Huang et al. 2003) *$\phi(\mathbf{x}) \geq j$ $(j = 1, 2, \cdots, M)$ if at least k_l consecutive components are in state l or above for all l $(1 \leq l \leq j)$. A system with such a structure function is called a multi-state consecutive-k-out-of-n:G system.*

In this paper, we focus on the following special case of this definition.

> *When $k_1 \geq k_2 \geq \cdots \geq k_M$, the system is called a decreasing multi-state consecutive-k-out-of-n:G system. In this case, for the system to be at a higher state j or above, a smaller number of consecutive components need to be at state j or above. In other words, as j increases, there is a decreasing requirement on the number of consecutive components that must be at state j or above for the system to be at state j or above.*

For this special system structure, Huang et al. (2003) observe the following properties of a decreasing multi-state consecutive-k-out-of-n:G system: (1) $n \equiv k_0 \geq k_1 \geq k_2 \geq \ldots \geq k_M$; (2) The minimal path vectors to system state j will cause the system to be exactly in state j; (3) One of the minimal path vectors to state j is in the following form:

$$(\underbrace{\underbrace{\underbrace{\underbrace{j, \ldots, j}_{k_j}, j-1, \ldots, j-1}_{k_{j-1}}, j-2, \ldots, 1, \ldots, 1}_{k_1}, 0, \ldots, 0}_{n}), \tag{1}$$

where the number of elements taking the value of i is equal to $k_i - k_{i+1} \geq 0$ for $i = 1, \ldots, j-1$; and (4) Every minimal path vector to state j can be obtained by permutating the elements of the minimal path shown in (1). Not all permutations of the vector in (1) qualify to be a minimal path vector to state j.

Let $\mathbf{y}$ be a minimal path vector to state j and assume that there are more than two different values in this minimal path vector. Define $s = \mathrm{Min}\{i | i \in \mathbf{y}, i < j\}$, $t = \mathrm{Max}\{i | i \in \mathbf{y}, i < j\}$, and $u_i = P_{ij}$ for $i = 1, 2, \ldots, n$. Note that we have $0 \leq s \leq t < j$. Now define

$$v_i = \begin{cases} 1 - u_i - Q_{is}, & \text{for upper bound calculation;} \\ 1 - u_i - Q_{it}, & \text{for lower bound calculation;} \end{cases} \qquad i = 1, 2, \ldots, n. \tag{2}$$

Huang et al. (2003) provides the following equation for bounding $\Pr(\phi \geq j)$.

$$R(n; k) \;=\; v_n R(n-1; k) + u_n \left[\left(\prod_{i=n-k+1}^{n-1} u_i \right) R^*(n-k) + \sum_{i=n-k+1}^{n-1} v_i \left(\prod_{l=i+1}^{n-1} u_l \right) R(i-1; k) \right], \tag{3}$$

where $R^*(i) \equiv \prod_{l=1}^{i}(u_l + v_l)$ for $i \geq 1$, $R(k;k) = u_1 u_2 \cdots u_k$, and $R(a;b) = 0$ for $b > a > 0$.

When the minimal path vectors have at most two different element values, equation (3) provides the exact measure of the probability for the system to be in state j or above.

3 State Distribution of a Decreasing Multi-State Consecutive-k-out-of-n:G System with only Three Possible States

In the following, we report an algorithm for evaluation of $\Pr(\phi \geq j)$ for a decreasing multi-state consecutive-k-out-of-n:G system when every minimal path vector to state j has exactly three different values, $\{m_0,\ m_1,\ m_2\}$, where $0 \leq m_0 < m_1 < m_2 = j$. One of the minimal path vectors will then have the following form:

$$\underbrace{(\underbrace{\underbrace{m_2, \ldots, m_2}_{k_2}, m_1, \ldots, m_1}_{k_1}, m_0, \ldots, m_0}_{k_0 = n}), \tag{4}$$

For the system to be in state j or above, at least k_2 consecutive components must be in state $m_2 = j$ or above, at least k_1 consecutive components must be in state m_1 or above, and all components must be in state m_0 or above. Based on earlier assumptions, we have $n = k_0 > k_1 > k_2$.

$$\begin{aligned} R_2(n, k_1, k_2) &= \Pr(\text{at least } k_l \text{ consecutive components are in state } m_l \text{ or above for } l = 0, 1, 2) \\ &= \text{Term}_1 + \text{Term}_2 + \text{Term}_3 + \text{Term}_4, \end{aligned} \tag{5}$$

$$\begin{aligned} \text{Term}_1 &= (Q_{nm_1} - Q_{nm_0}) R_2(n-1, k_1, k_2) \\ \text{Term}_2 &= \sum_{i=n-k_1+1}^{n-1} (Q_{im_1} - Q_{im_0}) \left[R(i-1, k_1) R^{(n)}(n-i, k_2) + R_2(i-1, k_1, k_2) \left(\prod_{j=i+1}^{n} P_{jm_1} - R^{(n)}(n-i, k_2) \right) \right] \\ \text{Term}_3 &= R^{(n)}(k_1, k_2) \prod_{i=1}^{n-k_1} P_{im_0} + R'(n-k_1, k_2) \prod_{i=n-k_1+1}^{n} P_{im_1} - R'(n-k_1, k_2) R^{(n)}(k_1, k_2) \\ \text{Term}_4 &= \sum_{h=1}^{k_2-1} [(\prod_{l=n-k_1+1}^{n-k_1+h} P_{lm_2})(Q_{(n-k_1+h+1)m_2} - Q_{(n-k_1+h+1)m_1})(\prod_{l=n-k_1+h+2}^{n} P_{lm_1} - R^{(n)}(k_1-h-1, k_2)) \\ &\quad \times (\prod_{l=n-k_1-k_2+h+1}^{n-k_1} P_{lm_2}) Q_{(n-k_1-k_2+h)m_2} (\prod_{i=1}^{n-k_1-k_2+h-1} P_{im_0} - R'(n-k_1-k_2+h-1, k_2)). \end{aligned}$$

where $R(a,b)$ can be calculated with equation (3) with the following input: $u_i = P_{im_1}$ and $v_i = Q_{im_1} - Q_{im_0}$ for $i = 1, 2, \ldots, a$; $R'(a,b)$ can be calculated with equation (3) with the following input: $u_i = P_{im_2}$ and $v_i = Q_{im_2} - Q_{im_0}$ for $i = 1, 2, \ldots, a$; $R^{(n)}(a,b)$ can be calculated with equation (3) with the following input: $u_i = P_{im_2}$ and $v_i = Q_{im_2} - Q_{im_1}$ for $i = n-a+1, n-a+2, \ldots, n$; and an empty summation represents zero while an empty product represents 1.

4 Bounding the State Distribution of a Decreasing Multi-State Consecutive-k-out-of-n:G System with More than Three Possible States

With the above algorithms for systems with only three distinct states, we now describe the algorithm for bounding system state distribution when there are more than three distinct states. Suppose we have a minimal path vector for system state j, denoted by $\mathbf{y}$, which is in the form shown in equation (1). We will use $\mathbf{y}_\mathbf{j}^*$ to represent all minimal path vectors to system state j. Then, we have $\Pr(\phi \geq j) = \Pr(\mathbf{x} \geq \mathbf{y}_j^*)$, where $\mathbf{x}$ represents all possible component state vectors. If $\mathbf{y}$ has more than three different element values, define

$$\mathbf{L} = (\overbrace{j, j, ..., j}^{k}; \overbrace{s, s, ..., s}^{n-k}), \quad \mathbf{U} = (\overbrace{j, j, ..., j}^{k}; \overbrace{t, t, ..., t}^{n-k}) \tag{6}$$

$$\mathbf{L}_1 = (\overbrace{j, j, ..., j}^{k};\ \overbrace{t, t, ..., t}^{k_{(t)}}; \overbrace{s_1, s_1, ..., s_1}^{n-k-k_{(t)}}), \quad \mathbf{U}_1 = (\overbrace{j, j, ..., j}^{k};\ \overbrace{t, t, ..., t}^{k_{(t)}};\ \overbrace{t_1, t_1, ..., t_1}^{n-k-k_{(t)}}) \tag{7}$$

where $s = \text{Min}\{i|i \in \mathbf{y}, i < j\}$, $t = \text{Max}\{i|i \in \mathbf{y}, i < j\}$, $s_1 = \text{Min}\{\{i\,|i \in y\} \setminus \{t\}\}$, $t_1 = \text{Max}\{\{i\,|i \in y\} \setminus \{t\}\}$, and $k_{(t)}$ is the number of t's in the minimal path vector $\mathbf{y}$. Obviously, we have $\mathbf{L} \leq \mathbf{L}^1 \leq \mathbf{Y} \leq \mathbf{U}^1 \leq \mathbf{U}$.

Let $\mathbf{L}^*$ represent all component state vectors in which exactly k consecutive elements have a value of j and all other elements have a value of s. Let $\mathbf{U}^*$ represent all component state vectors in which exactly k consecutive elements have a value of j and all other elements have a value of t. Let $\mathbf{L_1^*}$ represent all component state vectors in which exactly k consecutive elements have a value of j, exactly $k_{(t)}$ consecutive elements have a value of t, and all other elements have a value of s_1. Let $\mathbf{U_1^*}$ represent all component state vectors in which exactly k consecutive elements have a value of j, exactly $k_{(t)}$ consecutive elements have a value of t, and all other elements have a value of t_1. We then have

$$\Pr(\mathbf{x} \geq \mathbf{L}^*) \leq \Pr(\mathbf{x} \geq \mathbf{L}_1^*) \leq \Pr(\phi \geq j) \leq \Pr(\mathbf{x} \geq \mathbf{U}_1^*) \leq \Pr(\mathbf{x} \geq \mathbf{U}^*). \tag{8}$$

As a result, the bounds using $\mathbf{L_1}$ and $\mathbf{U_1}$ proposed in this paper are tighter than the bounds using $\mathbf{L}$ and $\mathbf{U}$, as reported by Huang et al. (2003).

For more details on the results reported in this paper, readers are referred to Zuo et al. (2003).

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